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ASPECTS OF FAIR DIVISION

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ABSTRACT

Provides an exposition of several elementary models of fair division as they occur in the theory of games, inclusive of: The Problem of the Nile, The Ham Sandwich Problem, and The Steinhaus Solution of Divide and Choose. The discussion is intended to be essentially nontechnical, relying mostly on heuristic methods of presentation, avoiding rigorous derivation as much as possible. A broad distinction is made between mathematical formulations and game theoretic, observing however, common inter-sections.

I. GENERAL DISCUSSION

> ...To unequals, equals become unequal if they are not harmonized by measure.
>
> Plato, Laws

In the history of social doctrine, philosophers and, on occasion, economists have agonized over the problem of defining the "just society," whose chief characteristic is the mutual satisfaction of its members with their designated lots. In The Leviathan of Hobbes, where reasonable men have contracted among themselves to avoid the "nasty, brutish and short" alternative consequence, we find the above characteristic expressed as a prima dictum:

> The first law is for the division of the land itself; wherein the sovereign assigneth to every man a position according as he himself shall judge agreeable to equity.
>
> Thomas Hobbes, The Leviathan

However one may be committed to this point of view, such dicta in themselves do not provide substantive criteria for actually carrying out such a programme. Indeed, the noted nineteenth century ethicist-economist John Stuart Mill, while seemingly convinced of the objective necessity and essential correctness of the laws of production in governing the efficient allocation of goods, could only turn a cynical cheek to the problem of formulating criteria of similar ilk to govern the initial distributions of wealth in an equitable manner:

> The distribution of wealth is a matter of human intuition solely. The things once there, mankind, individually or collectively, can do with them as

> they like. They can place them at the disposal of whomsoever they please, and on whatever terms.... It therefore depends on the laws and customs of society. The rules by which it is determined are what opinions and feelings of the ruling portion of the community make them, and are very different in different ages and countries, and might be still more different, if mankind so chose.
>
> John Stuart Mill, Principles of Political Economy

For better or worse, game theorists with the assistance of their mathematical confreres have been undaunted by Mill's cynicism and have sought to deal with the above problem in a formal manner through the use of models that are mathematical in nature. Reformulated slightly, the classical problems concerning the existence of a just society* are expressed as games of fair division, which in some instances yield interesting results, as we hope the discussion that follows will indicate.

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*The recent treatise of John Rawles, A Theory of Justice, Harvard University Press, 1974, explores such a thesis in detail.

II. AN HISTORICAL REVIEW OF DIVISION PROBLEMS

In this section, we provide something in the way of background for what is to follow by considering two varieties of "division" problems: mathematical and game theoretic. The two denominations are by no means disjoint for, as is well known, solutions in one very often provide solutions in the other classification. The distinctions are for the purpose of indicating the origins of the results in the literature.

MATHEMATICAL FORMULATIONS

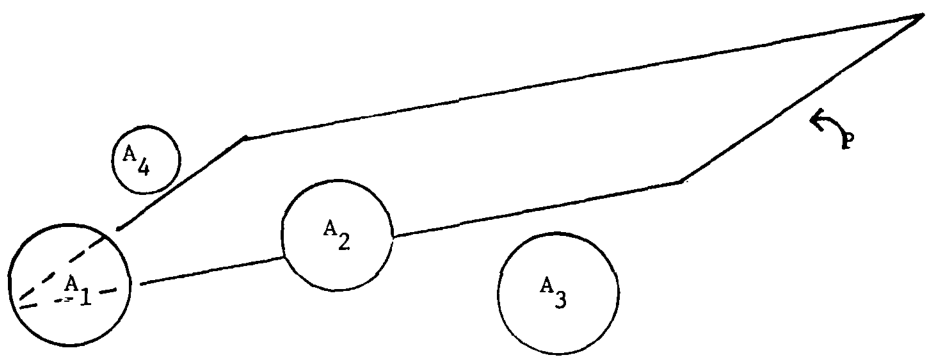
The Ham Sandwich Problem

The original version of this problem was formulated by the Polish mathematician S. Ulam, in the 1930s [13]. A short time later Hugo Steinhaus, also of the Polish school, gave the problem its nickname. The problem can be stated as follows: Given three finite and arbitrary measures (volumes): A_1, A_2, and A_3, in three dimensional space, is it possible to find at least one plane P, also in three dimensional Euclidean space, that will simultaneously bisect each of the three volumes? (The key condition is that the spheres are arbitrarily placed.) Thus, one could easily visualize a ham sandwich composed of arbitrarily distributed ham, cheese and bread: the problem being to slice with a knife the sandwich into two portions containing an equal amount of each ingredient.

Because of the topological invariance of a dimension, the above problem must be restricted to the case where the number of objects (measures) to be divided does not exceed the number of dimensions of the space in question, and where the dividing hyperplane is of the next lower dimension. Thus one can simultaneously bisect up to three finite volumes in three dimensions by a

two dimensional plane. Inductively speaking, one can then pose the generalized Bisection Problem as: Is it possible to simultaneously bisect each of n bodies in n-dimensional Euclidean space by an n-1 dimensional hyperplane? In 1942, A. H. Stone along with J. W. Tukey provided an affirmative answer to the Generalized Bisection Problem [12].

That these results fail if the restrictions of dimension are not observed can be seen with the help of some elementary geometry:



The plane P, bisecting A_1 and A_2, can be "rotated" so as to also bisect either A_4 or A_3, but not both simultaneously, in addition to bisecting A_1, and A_2.

The Problem of the Nile

The original version of this problem was given by the statistician R. A. Fisher around 1937 [6]. A slightly modified account of Fisher's version is the following:

> Each year the Nile would overflow its banks and thereby irrigating or perhaps devastating parts of the agricultural flood plain of a pre-dynastic Egyptian province. The values of different portions

> of the land would therefore depend on the level of the flood. Is it possible for the magistrate to sub-divide the land in a manner that would guarantee to each farmer 1/k th of the total value of the land irrespective to the possible levels of the flood?

This problem also bears close relation to the problem of similar regions defined by Neyman and Pearson [9]. Indeed, under quite natural conditions, the two problems have been proved to be equivalent by Darmois in 1946 [2].

GAME THEORETIC

Equal Treatment Games

Loosely stated, these are games the solutions of which treat equal participants equally. Equality among participants can be defined in terms of "bargaining strength," for example, an initial endowment configuration along with given tastes, or the amount a participant contributes when he joins a coalition. In all versions however, the extent to which a participant is entitled to a share is governed by his relative position at the commencement of the game. We now turn to some notable examples of such games.

The Shapley Value Outcome

Given an n-person cooperative game in characteristic function form, (N,V), where N is the number of participants and $v: 2^N \to \mathbb{R}_+$, the Shapley value outcome is given by the formula: (L. S. Shapley [11])

$$\forall i \in N \quad \Phi_i = \sum_{s \in 2^N} (s-1)!\,(n-s)!/_{n!}\,[v(s) - v(s-\{i\})]$$

where Φ_i is the Shapley valuation of the ith participant. If the game (N,v) is symmetric in the Von Nuemann-Morgenstern sense ([14], Ch.X), it can be shown that $\Phi_1 = \Phi_2 = \Phi_3 = \ldots = \Phi_n$. However, it should be pointed out that

the assumption that the game be symmetric in a sense presupposes equal treatment. The Shapley value outcome must then be viewed as a restatement of an assumed property in this instance. Our next two examples demonstrate that the outcomes themselves, can in some instances, exhibit equal treatment or symmetry, rather than presuppose it.

Competitive Market Mechanisms and the Core

Consider a pure exchange economy, where each participant is characterized by what Debreu ([3], Ch.I) calls a complete pre-order $\leq_i$ and a set of endowments, represented by a point in n-dimensional Euclidean space, $\omega_i \varepsilon E^n_+$. Assume there are M participants, M being some initial segment of the positive integers which serves to index them. The following assumptions will be required for all $i \varepsilon M$.

(i) Insatiability: If χ is an arbitrary commodity vector in E^n_+, then there is a commodity vector χ' in E^n_+ such that $\chi <_i \chi'$.

(ii) Strong Convexity: If χ and χ' are two different commodity bundles such that $\chi \leq_i \chi'$, then for $\alpha \varepsilon (0,1)$, $\chi <_i \alpha\chi + (1-\alpha)\chi'$.

(iii) Continuity (of the Cantor variety): For a commodity vector $\chi' \varepsilon E^n_+$, the following sets are closed: $\{\chi : \chi \leq_i \chi'\}$ $._\wedge.$ $\{\chi : \chi' \leq_i \chi\}$.

(iv) Strict Positivity of Individual Resources: For each participant, the vector ω_i is strictly interior to E^n_+, i.e., $\omega_i >> 0$.

We now define the notion of the core. Among all feasible allocations, $(\chi_1, \ldots, \chi_m) \varepsilon E^{nm}$ such that $\sum_{i \varepsilon m} (\chi_i - \omega_i) = 0$, the core is that set of allocations that cannot be "blocked." An allocation is said to be blocked if there is some subset of participants S, who can improve the position of at least one member of S, at no expense to any other member, by redistributing their

own resources. More formally, $(\chi_1, \ldots, \chi_m)$ is in the core if $\sum_{i\varepsilon M} (\chi_i - \omega_i) = 0$, and for any other allocation $(\chi'_1, \ldots, \chi'_m)$ such that $\sum_{i\varepsilon M} (\chi'_i - \omega_i) = 0$ for no subset $S\varepsilon 2^m$ can $\sum_{i\varepsilon S} (\chi'_i - \omega_i) = 0$ and $\chi_i \leq_i \chi'_i$ for all $i\varepsilon S$ with $\chi_j <_j \chi'_j$ for at least one $j\varepsilon S$, hold simultaneously.

From the above assumptions, (i)-(iv), it can be shown that a competitive equilibrium exists, i.e., there are vectors: $\hat{\chi} = (\hat{\chi}_1, \ldots, \hat{\chi}_m)$ and $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)$ such that $\sum_{i\varepsilon M} (\chi_i - \omega_i) = 0$ and $\hat{\chi}_i$ is a maximal element for $\leq_i$ over the set characterized as $\{\chi_i : \hat{p}\cdot\chi_i \leqq \hat{p}\cdot\omega_i\}$.

Theorem: A competitive allocation is in the core.

Pf: (Debreu-Scarf [4]) Any blocking subset S where $\sum_{i\varepsilon S} (\chi'_i - \omega_i) = 0$ while $\chi_i \leq_i \chi'_i$ for all $i\varepsilon S$ while $\chi_j <_j \chi'_j$ for some $j\varepsilon S$, would entail $\sum_{i\varepsilon S} \hat{p}\cdot\chi'_i > \sum_{i\varepsilon S} \hat{p}\cdot\hat{\chi}_i$, which in turn implies $\sum_{i\varepsilon M} \hat{p}\cdot\chi'_i > \sum_{i\varepsilon M} \hat{p}\cdot\omega_i$ which contradicts the feasibility of the allocation $(\chi'_1, \ldots, \chi'_m)$.

Next, if we were to replicate the above economy to R consumers of each type, of which previously there were m participants, an allocation can then be characterized as an array of MR commodity vectors, χ_{iq}, which denotes the q^{th} individual's allocation of the i^{th} type. The allocations that are feasible must satisfy: $\left(\sum_{i\varepsilon M} \sum_{q\varepsilon R} \chi iq - R \sum_{i\varepsilon M} \omega_i\right) = 0$.

Theorem: An allocation in the core assigns the same consumption to all consumers of the same type.

PF: (Debreu-Scarf [4]) For a given type i, let χ_i denote the least desired of the feasible vectors χ_{iq}, and assume that for some type i' some two consumers have been assigned different commodity bundles. Apparently, $\chi_i \leq_i 1/R \sum_{q\varepsilon R} \chi_{i'q}$ whereas for i' we have $\chi_i < 1/R \sum_{q\varepsilon R} \chi_{i'q}$; the latter by the assumption of Strong Convexity. However we have a contradiction because feasibility requires that $\sum_{i\varepsilon m} (1/R \sum_{q\varepsilon R} \chi_{iq} - \omega_i) = 0$.

Non-Cooperative (Cooperative) Fair Division - The Non-atomic Case

This problem was first formulated and solved for the case where the number of participants is two, by Hugo Steinhaus. The object of the game is to obtain a partition of an object which is infinitely divisible, like a cake whose granules are infinitesimally small, and assign a portion to the participants which satisfy their individual estimates of a "fair share." The procedure is for one person to partition the cake into two equal portions, according to his own estimation, while the second person chooses a portion which satisfies his own criterion of fairness. Thus in a certain sense, the advantage is to the chooser because the one who divides does not have an opportunity to receive more than half of the "cake," according to his own measure, unless he is willing to incur risk of receiving less than half. The chooser, on the other hand, may have a chance to receive more than his fair share without incurring a similar risk. This imperfection of role-advantage can be removed from the game if the procedure is formalized as an extensive game in which the first move is an equiprobable determination of the role of chooser. In fact, this is the approach of Kuhn [8], which we will treat below.

The above procedure, which we shall call the classical scheme, was intended by the Polish mathematical team of Banach and Knaster [7]. Conceptually, their solution was the following: a knife is slowly, but continuously moved over the "cake" parallel to itself. At each instant in time, the knife is positioned to cut a unique portion of the cake. The first person to indicate satisfaction with the slice then determined by the position of the knife calls out to stop the process and is awarded that share of the cake. If two participants call out simultaneously, again an equiprobable device is called into play in order to determine who gets the share.

This latter procedure is "fair" in the sense that if each participant were to adope the strategy of calling out his satisfaction at that position of the knife which would cut a portion of the cake equal to 1/n the valuation of the cake, in his eyes, either (1) the participant is given the first piece of the cake which is 1/n of the total according to his tastes, or (2) the participant becomes one of the n-1 participants who have a share in the remaining portion of the cake which is worth at least (n-1)/n of the original cake according to his own evaluation. Inductively, each participant must receive at least 1/n of the total value according to his own evaluation.

The significance of this result is more than that of existence, for it provides an effective procedure for carrying out a programme such as Hobbes' and enjoys the desirable mathematical properties of uniqueness and stability. Furthermore, in this approach participants are entitled to their fair shares on the basis of their participation solely. No restriction is placed on their ability to contribute in a "productive" sense to the group considered in its entirety.

One serious limitation of the approach, however, is that the nature of the framework specifies the allocation of an effectively non-atomic object in a measure theoretic sense. However, in order to be on safe ground in measure theory, one must assume that non-atomic objects be measurable in a sense that could be restrictive.

The Allocation of n Objects to K Participants or Non-Cooperative (Cooperative) Fair Division - The Atomic Case

Historically, the first recorded instance of this problem is "Solomon's Dilemma": the problem of awarding a baby to its rightful mother, whom two candidates claimed virgorously to be. Momentarily pretending that the baby

was a non-atomic entity, Solomon threatened to end the dispute by enforcing the classical scheme of divide and choose and was thereby able to perceive the true mother's identity. The generalized fair division scheme, also from Banach and Knaster [7], can be viewed as an extension of Solomon's wisdom in this particular instance. Their solution in the general case of n participants and k objects was for each participant to make a bid on each object. Each object is then assigned to its highest bidder who then must pay a surplus tax to the other participants which will insure an equal amount of surplus over the "fair share" for each participant.

This approach is less general in structure than in the case which the object to be divided is finely divisible. Restrictions must be placed on the amount each participant can bid lest some one buy the pot. Additionally, this procedure does not seem to avoid Plato's objection concerning the equitable nature of an equal amount of surplus to participants who are unequal in tastes, by way of normalizing shares in money terms. All of which should indicate the difficulty of formulating an effective procedure where indivisibilities exist. For more details on the atomic case, the reader should consult Dubins and Spanier [5].

III. MATHEMATICAL TREATMENT OF SPECIFIC PROBLEMS

Generally, one may classify the mathematical treatments as being either computational or existential. The former variety of analysis is particularly concerned with effectively recursive and self-enforcing schemes that indicate how a fair division can actually be carried out by the participants themselves. The latter variety is concerned with specifying the conditions under which a fair division is possible. The same distinction can be made with respect to competitive equilibrium theory between existence and stability. (Cf. Scarf [10], and Arrow and Hahn [1].)

Specifically, we treat in an existential manner the following:

(1) The General Bisection Problem

(2) The Problem of the Nile

and provide a formalization of Steinhaus's classical scheme of Divide and Choose and present H. W. Kuhn's extension of that scheme to the general case of n participants.

The General Bisection Problem

Essential to the proof of the existence of a solution to the General Bisection Problem is the following theorem of Borsuk and Ulam:

Borsuk-Ulam Theorem: Let f be a continuous mapping of the n-dimensional sphere S^n into E^n such that $f(\chi) + f(-\chi) = 0$ for every $\chi \varepsilon S^n$, i.e., f is antipodal. There then exists an $\hat{\chi} \varepsilon S^n$ such that $f(\hat{\chi}) = 0$.

Pf: The proof, although not complicated is somewhat advanced and I prefer to refer the reader to Lefshetz's Introduction to Topology, Princeton University Press, 1951, for an excellent version.

We can now prove:

Theorem: Let Ω-algebra of Borel subsets of S^n, i.e., the measurable subsets of S^n. Let ω be the usual rotation invariant measure on Ω, and let $\omega_1, \ldots, \omega_n$, be n countably additive real valued measures on Ω each absolutely continuous with respect to ω. There then exists a closed hemisphere E of S^n, such that for all i, $\omega_i(E) = \omega_i(S^n)/2$.

Pf: For each $\chi \varepsilon S^n$, let $E(\chi)$ be the closed hemisphere of all $\gamma \varepsilon S^n$ whose inner product with χ is non-negative. Then, $E(\chi) \cup E(-\chi) = S^n$ and $\omega(E(\chi) \cap E(-\chi)) = 0$.

If χ and χ' are close points of S^n, then the symmetric difference of $E(\chi)$ and $E(\chi')$ has small ω-measure. Since each ω_i is absolutely continuous with respect to ω, it also has small measure according to ω_i. Therefore, for each i, the function $\omega_i(E(\chi))$ is a continuous function on S^n.

Define f on S^n by the following:

$$f(\chi) = (\omega_i(E(\chi) - \omega_1(S^n)/2, \ldots, \omega_n(E(\chi)) - \omega_n(S^n)/2)$$

Then f is a continuous mapping of S^n into Euclidean n-space such that $f(\chi) + f(-\chi) = 0$, for all $\chi \varepsilon S^n$. The Borsuk-Ulam Theorem then entails the existence of an $\hat{\chi} \varepsilon S^n$ such that $f(\hat{\chi}) = 0$. From the construction, $E(\hat{\chi})$ satisfies the condition that $\omega_i(E(\hat{\chi})) = \omega_i(S^n)/2$ or all i.

Corollary: Let $E_1, \ldots, E_n$ be measurable subsets of S^n. Then there exists a hemisphere that contains half of each E_i.

Pf: Let $\omega_i(E_i):\omega_i(S^n) - \omega(E_i) = \omega_i(S^n)$. By the above theorem, there exists an $E(\hat{\chi}):\omega_i(E(\hat{\chi})) = \omega(E_i)/2$ for each E_i.

If we interpret a hyperplane of n-1 dimensions as a hemisphere of infinite radius in E^n, both the theorem and its corollary provide an affirmative answer to the question posed by the General Bisection Problem.

The Problem of the Nile

The problem, as formulated in the second section of this paper, has a solution provided that the possible levels of flood are finite in number. The proof of existence rests upon a theorem of Liapunov, which asserts that the image set of a vector valued measure is convex.

Let U denote the flood plain and let A_U denote a σ-algebra of subsets of U, that is to say $\phi\varepsilon A_n$, $U\varepsilon A_U$, $\beta\varepsilon A_U \Rightarrow \beta^c \varepsilon A_U$, and $\beta_1, \ldots, \beta_n, \ldots,$ $\varepsilon A_U \Rightarrow \bigcup_j \beta_j \varepsilon A_U$.

Consider next n normalized, countably additive real-valued measures defined on A_U, $U_1, \ldots, U_n$; the sets of A_U are assumed to be measurable in terms of the $U_1, \ldots, U_n$, and the $U_1, \ldots, U_n$ are such that $U_i(U) = 1$, $\forall i\varepsilon N$.

Denote by δ, the set of ordered partitions of U:

$P\varepsilon\delta \Leftrightarrow \{P_1, \ldots, P_k\} = P$ and $\bigcup_j^K P_j = U$ and $P_j \cap P_i$ if $j \neq i$.

For a member $P\varepsilon\delta$ (of order K), we define the mapping: $M:\delta \to E^{nK}$ where M(P) is an n x K matrix. Denote by R the set of matrices corresponding to M(δ), the image set of δ under the mapping M.

We state Liapunov's theorem without proof, as it is well known.

Theorem (Liapunov): If the $U_1, \ldots, U_n$ are non-atomic, R = M(δ) is convex.

Let $\{\alpha_j\}$ denote a set of rational numbers such that $\sum_j \alpha_j = 1$. Assume that the α_j are strictly positive as well.

Theorem: If each U_i is non-atomic, given $\alpha_1, \ldots, \alpha_K$ such that $\sum_j \alpha_j = 1$, there exists a partition $\hat{P}\varepsilon\delta$ of U such that $U_i(\hat{P}_j) = \alpha_j$ for all $i\varepsilon I = \{1, \ldots, n\}$ and $j = 1, \ldots, K$.

Pf: Let P_j be the partition of U in δ which assigns the entire set U to the j^{th} member of the partition. Obviously,

$$M(P_j) = \left\| \begin{matrix} 0 & \dots & 1 & 0 & \dots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & \dots & 1 & 0 & \dots & 0 \end{matrix} \right\|$$

where the 1's occur in the j^{th} column.

By Liapunov's Theorem, the matrix defined by $\Sigma_j \alpha_j M(P_j)$ is in $R = M(\delta)$ since $M(\delta)$ is the range of the matrix mapping $M:\delta \rightarrow E^{nK}$, there exists at least one partition $\hat{P}$ in δ such that the j^{th} column of $M(\hat{P})$ equals the j^{th} column of $\Sigma\alpha_j M(P_j)$. Then for $M(\hat{P})$, $U_i(\hat{P}_j) = \alpha_j$ for all $i\varepsilon I$.

If we identify the number of participants with K and the n measures as evaluations of the flood plain under differing levels of floods, the preceding result yields an affirmative answer to the question posed in the Problem of the Nile. The method of proof we have employed is found in Dubins and Spanier [6].

Kuhn's Formulation of Steinhaus's Classical Scheme

By a fair division problem will be meant the following: (S, N, δ, F) where S is some set to be divided among the participants, N is the number of participants, δ is a family of partitions of order N, (i.e., $P\varepsilon\delta$ and only if $P = \{\tilde{S}_1, \dots, \tilde{S}_n\}$ and $\bigcup_j S_j = S$ while $\tilde{S}_j \cap \tilde{S}_i = \phi$ if $j \neq i$), and F is a family of classes consisting of subsets of S. For the latter set, we have $F_i \varepsilon F$ if and only if all sets $T\varepsilon F_i$ where $T \subseteq S$ are acceptable as the i^{th} participant's fair share.

By a legal division will be meant an assignment of the sets $\tilde{S}_j$ of a partition $P\varepsilon\delta$, to the players N. If S_{ji} is assigned to player i and if $S_{ji}\varepsilon F_i$ for

all $i = 1, \ldots, N$, then a legal division is said to be a fair division. For the purpose of depicting an assignment, we will employ the following notation: $\chi = (\chi_{ij})$ where $\chi_{ij} = 1$ if $S_j \rightarrowtail i^{th}$ participant and $\chi_{ij} = 0$ otherwise (:f$S_j \rightarrowtail i^{th}$ participant, then S_j is assigned to the i^{th} participant).

We turn next to a formalization of the classical scheme in terms of an extensive game as given by Kuhn [8]. For purposes of illustration, let S be a "cake," $N = \{1,2\}$, and $F = \{F_1,F_2\}$

Move 1: The players are assigned the roles of chooser and divider by an equiprobable device

Move 2: The divider selects $P = \{\tilde{S}_1,\tilde{S}_2\}$ such that $P\varepsilon\delta$

Move 3: The chooser selects $\chi = (\chi_{ij})$ such that $\sum_i \chi_{ij} = \sum_j \chi_{ij} = 1$

Payoff: Player receives share S_j if and only if $\chi_{ij} = 1$

To insure the "fairness" of the scheme we need only assume

(a) No matter how the cake is divided legally, each player will find at least one of the pieces acceptable to him.

(b) Either player is able to divide the cake so that both pieces are acceptable to him.

The following is Steinhaus' solution for the case of three participants, expressed in Kuhn's formalism. Let S be as before, and let $N = \{1,2,3\}$ and $F = \{F_1,F_2,F_3\}$.

Move 1: The players are assigned roles of divider and chooser by means of an equiprobable device.

Move 2: The divider selects $P = \{\tilde{S}_1,\tilde{S}_2,\tilde{S}_3\}$.

Move 3: Each chooser announces which of the portions is acceptable to him, i.e., which $S_j \varepsilon F_i$ for each i.

Move 4: If a share can be assigned to each of the non-dividers in an acceptable way, this is done and the divider is assigned the remaining portion. If this is not the case, then some share is unacceptable to both non-dividers. This share is given to the divider and the remaining participants perform the classical two person scheme.

To insure the fairness of the scheme, we need only assume:

(a) For any partition, each player will find at least one of the pieces acceptable to him.

(b) Each player can divide the cake so that each of the three pieces is acceptable to him.

(c) If a piece is found unacceptable to both non-dividers, then the remaining portion, after that piece has been assigned to the divider, is considered to be a fair amount to divide among themselves.

We now turn to a formal scheme of Banach and Knaster's extension of the classical scheme to the general n-person case. However, we are first in need of more preliminary definitions.

Let $T \subseteq S$ and $M \subseteq N$ contain M participants. The restriction of δ to T and M will be denoted by $\delta(T,M)$ and consists of all partitions $\{S_1, \ldots, S_m\}$ of T such that there exists $S_{m+1}, \ldots, S_n$ together with the latter, forming a member of δ.

Suppose $P = \{S_1, \ldots, S_n\} \varepsilon \delta$ is such that $\bigcup_j S_j = T$ and $S_j \notin F_i$ for $i \varepsilon M$ and $M+1 \leq j \leq n$. Then $\delta(T,M)$ is called a fair restriction of δ to T and M. Notice that this entails, by definition, that $\delta(N,S)$ is a fair restriction. In

addition, we assume that all fair restrictions $\delta(T,M)$ of δ to T and M satisfy:

(i) For all $i\varepsilon M$ and $P\varepsilon\delta(T,M)$, $P\cap F_i \neq \phi$

(ii) For all $i\varepsilon M$ there is a $P_i\varepsilon\delta(T,M)$ such that $P_i \subset F_i$

i.e., each player finds at least one of the portions acceptable to him in a fair restriction and any player can subdivide a fair restriction in a manner that all portions are acceptable to him.

Given an nxn matrix $A = (a_{ij})$ with $a_{ij} = 0$.v. 1, and assignment is a set $\{(i_{1,ji}), \ldots, (i_{R,jR})\}$ where $I = \{i_1, \ldots, i_R\}$ and $J = \{j_1, \ldots, j_R\}$ where $R \geqq 1$ and $a_{ij} = 1$ for $i\varepsilon I$ and $j\varepsilon J$. An assignment is said to be complete if $R = N$.

We also have need of the following combinatorial lemma, which we do not prove. A proof can be found in Kuhn's article [8].

Lemma: Let $A = (a_{ij})$ be an mxn matrix with entries 0 or 1 such that $a_{ij} = 1$ for $j = 1, \ldots, n$ and $\sum_j a_i \geqq 1$, such that $a_{ij} = 0$ for $i\varepsilon I$ and $j\varepsilon J$.

We are now prepared for our last principal result, the Banach-Knaster extension to the n-person case.

Let player 1, of the n participants, be designated as divider by an equiprobable device. Player 1 then selects $P = \{S_1, \ldots, S_n\}$ such that $\forall_j$ $S_j \varepsilon F_1$. This is possible by (ii). Define $A = (a_{ij})$ by $a_{ij} = 1$ if $S_1 \varepsilon F_i$ and $a_{ij} = 0$ otherwise. The matrix A then satisfies the hypothesis of the lemma, for $\sum_j a_i \geqq 1$ for $i = 2, \ldots, N$ by assumption (i). Hence, there exists an assignment $\{(i_1,j_1), \ldots, (i_r,j_r)\}$ with $I = \{i_1, \ldots, i_r\}$, $J = \{j_r, \ldots, j_r\}$ and $S_j \varepsilon F_i$ for $i\varepsilon I$ and $j\varepsilon J$.

Now assign S_j to player i for $i\varepsilon I$ and $j\varepsilon J$. Then if $T = \bigcup_{j\notin J} S_i$ and

$M = \{i : i \notin I\}$, the restriction to T and M of δ is a fair restriction and the process may be reiterated. Since $r \geq 1$, at least one player receives a part $S_j \varepsilon F_i$ at each iteration. The scheme obviously terminates for a finite number of participants. Futhermore $S_j \succ\!\!\rightarrow i$ only if $S_j \varepsilon F_i$ and hence the scheme is fair.

Examples of pathology with respect to the above scheme, and a discussion of open problems are to be found at the conclusion of Kuhn's article.

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